

# Tunable quantum interference: How to make morning, noon, and afternoon states

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## Abstract

We show that the  $N$ -photon states produced by interference between laser light and downconverted light at the input of a two path interferometer can be characterized by a single tuning parameter that describes a transition from phase squeezing to nearly maximal path entanglement and back. The quantum states are visualized on a sphere using the analogy between  $N$ -photon interference and the spin- $N/2$  algebra.

## 1 Introduction

The sensitivity of quantum phase measurements is limited by the quantum fluctuations in the two mode  $N$ -photon statistics of the light field states used to probe the phase shift [1]. If coherent input light is used, the sensitivity is limited by the shot noise in the seemingly random photon detection events, resulting in the standard quantum limit of  $\delta\phi^2 = 1/N$ . However, quantum coherence can decrease this phase estimation error up to the fundamental Heisenberg limit of  $\delta\phi^2 = 1/N^2$ . The  $N$ -photon state that achieves this maximal phase sensitivity is the superposition state of the state where all photons are in one arm of the interferometer and the state where all photons are in the other arm,

$$|\text{NOON}\rangle = \frac{1}{\sqrt{2}}(|N;0\rangle + |0;N\rangle). \quad (1)$$

In recognition of the technological challenges involved in trying to realize such states for high photon numbers, Jonathan Dowling has dubbed these states ‘high NOON’ state, and by now,

these states are commonly referred to as noon states in the literature [2].

There have been numerous proposals for the generation of noon states [3], leading up to the experimental generation of three and four photon states showing the expected  $N$ -photon coherence [4]. Unfortunately, the rather inefficient post-selection methods used in these experiments result in low visibilities and make it difficult to achieve even higher photon numbers. However, help may be on the way. As we recently discovered, high fidelity noon states can be generated by simply interfering downconverted photon pairs and laser light [5]. Experimentally, this technique has been pioneered by Lu and Ou, and its application to three photon noon state generation was proposed by Shafiei and coworkers from the same group. However, it was thought that the generation of higher photon number noon states is not possible with this method, because a fidelity of 100 % can only be obtained if additional non-linear elements are used [6]. In fact, the quantum interference between laser light and downconverted light generates a slightly squeezed noon state, indicating that the non-classical states generated by this method combine squeezing effects with multi-photon coherences.

In our recent research, we have studied the continuous transition from gradual phase squeezing when most photons originate from the laser [7] to multi-photon quantum coherences resulting in a maximal noon state fidelity of about 94% when the average number of photons from the laser and from the down-conversion is  $N/2$  each [5]. In the following, we illustrate the complete

transition from squeezing to noon state and back on the sphere defined by the spin- $N/2$  algebra of  $N$ -photon two mode states. It turns out that the resulting images suggest a new motivation for the noon state terminology, since the noon state is illustrated by a superposition of a state at the zenith of the upper half of the sphere and its mirror image in the lower half. Consequently, we can extend the terminology to include dawn states (where non-classicality begins as a slight elongation of a state at the ‘eastern’ horizon), morning states (where the squeezed state has separated from the horizon into a superposition within the ‘eastern’ half of the sphere), afternoon states (where the states have passed the zenith and moved to the ‘western’ half of the sphere), and evening states (where the states meet again at the ‘western’ horizon and non-classicality recedes).

## 2 Analogy between $N$ -photon interference and spin- $N/2$ rotations

Optical quantum phase measurements can be realized using a two path interferometer such as the Mach-Zehnder interferometer shown in fig. 1. An  $N$ -photon state (or the  $N$ -photon component of an arbitrary field state) can then be expressed in terms of the  $N+1$  level system defined by the two mode Fock states  $|N-n; n\rangle$ . This Hilbert space is equivalent to that of a spin- $N/2$  system, and the corresponding spin components can be expressed in terms of the input modes  $\hat{a}$  and  $\hat{b}$  using the Schwinger representation,

$$\begin{aligned}\hat{J}_1 &= \frac{1}{2}(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}) \\ \hat{J}_2 &= \frac{1}{2}(\hat{a}^\dagger \hat{b} + \hat{a} \hat{b}^\dagger) \\ \hat{J}_3 &= -\frac{i}{2}(\hat{a}^\dagger \hat{b} - \hat{a} \hat{b}^\dagger).\end{aligned}\quad (2)$$

As indicated in fig. 1, the three orthogonal components of this vector can be interpreted as half of the photon number differences between the input modes ( $\hat{J}_1$ ), between the output modes at

$\phi = 0$  ( $\hat{J}_2$ ), and between the paths of the interferometer ( $\hat{J}_3$ ). A phase shift is then equal to a rotation around the  $J_3$ -axis, given by the unitary transformation  $\hat{U}(\phi) = \exp(-i\phi\hat{J}_3)$ .

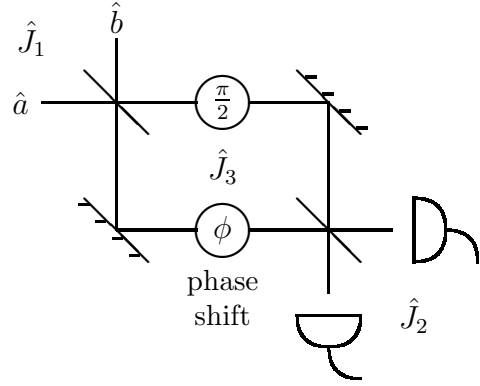


Figure 1: Illustration of two path interferometry using a Mach-Zehnder interferometer. The components of the Schwinger representation correspond to half the photon number differences at the input ( $\hat{J}_1$ ), at the output for  $\phi = 0$  ( $\hat{J}_2$ ), and inside the interferometer ( $\hat{J}_3$ ).

## 3 Interference between coherent laser light and down-converted light

As Ou and coworkers have pointed out [6], it is possible to generate  $N$ -photon quantum interferences by mixing coherent laser light and down-converted light at a beam splitter because the output measurement does not distinguish between photons from the laser and photons from the down-conversion. In their theoretical treatment, Ou and coworkers used the Fock state representations to derive the relation between the input and the output coherences. However, we found that a more compact representation of the coherences can be obtained by using the operator relations that characterize the coherent state  $|\alpha\rangle$  in mode  $\hat{a}$  and the single mode down-converted state  $|\gamma\rangle$  in mode  $\hat{b}$ ,

$$\hat{a} |\alpha\rangle = \alpha |\alpha\rangle$$

$$\hat{b} | \gamma \rangle = -\gamma \hat{b}^\dagger | \gamma \rangle. \quad (3)$$

The quantum coherence of the  $N$ -photon component  $| \eta \rangle$  of the two mode state can then be characterized by photon number preserving combinations of creation and annihilation operators for the two modes [5]. The most compact relation reads

$$\begin{aligned} \hat{a}^\dagger \hat{b} | \eta \rangle &= \eta \frac{\hat{a}^\dagger \hat{a}}{N} \hat{b}^\dagger \hat{a} | \eta \rangle \\ \text{where} \quad \eta &= \frac{N\gamma}{\alpha^2}. \end{aligned} \quad (4)$$

Thus the two mode  $N$ -photon state  $| \eta \rangle$  is defined by the single tunable parameter  $\eta$ , which is given by  $N$  times the ratio of down-conversion pair amplitude  $\gamma$  and squared coherent amplitude  $\alpha$ .

We can now transform relation (4) into a non-linear squeezing relation of the Schwinger parameters. The result reads [7]

$$\begin{aligned} \left( 1 + \left( \frac{1}{2} + \frac{\hat{J}_1}{N} \right) \eta \right) \hat{J}_2 | \eta \rangle &= \\ i \left( 1 - \left( \frac{1}{2} + \frac{\hat{J}_1}{N} \right) \eta \right) \hat{J}_3 | \eta \rangle. \end{aligned} \quad (5)$$

It is possible to derive the approximate features of the quantum state  $| \eta \rangle$  by appropriate linearizations of this squeezing relation.

## 4 Morning states: from linear squeezing to path entanglement

For  $\eta < 1$ , most of the photons originate from the laser light input, so  $\langle \hat{J}_1 \rangle \approx N/2$ . The linearized squeezing relation then represents a gradual redistribution of quantum noise from  $\Delta J_2$  to  $\Delta J_3$ . Specifically,

$$(1 + \eta) \hat{J}_2 | \eta \rangle \approx i(1 - \eta) \hat{J}_3 | \eta \rangle, \quad (6)$$

so the ratio of uncertainties in  $\hat{J}_2$  and  $\hat{J}_3$  is given by a squeezing factor of

$$\exp(2r) = \frac{\Delta J_3}{\Delta J_2} \approx \frac{1 + \eta}{1 - \eta}. \quad (7)$$

A representation of this squeezed state is shown in fig. 2. Since the state is still centered around the ‘eastern’ horizon at  $\mathbf{J} = (N/2; 0; 0)$ , we might call this the dawn state of our quantum interference scheme.

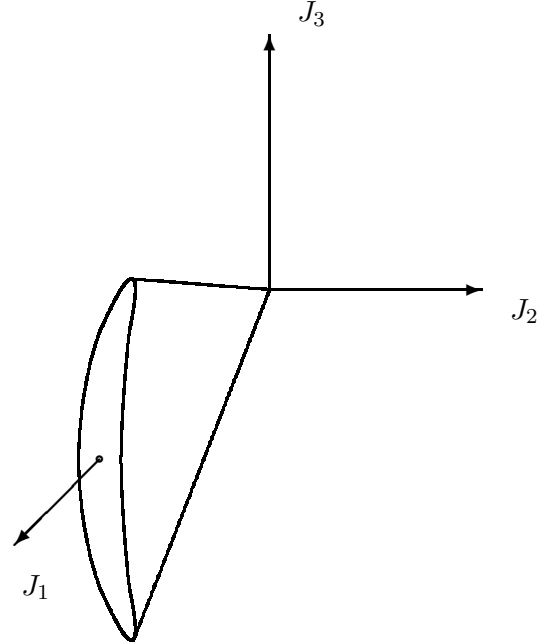


Figure 2: Dawn state ( $\eta < 1$ ). The addition of a small amount of down-converted photon pairs causes phase squeezing.

At  $\eta = 1$ , there is a transition between squeezing and a non-classical superposition of two separate regions on the  $\mathbf{J}$ -sphere. As we discuss in detail in [7], this transition results in a maximal phase squeezing of about  $\delta\phi^2 = 1/N^{3/2}$  near  $\eta = 1$ , corresponding to the geometric mean of standard quantum limit and Heisenberg limit. For  $\eta > 1$ , the state splits into a superposition of two  $\mathbf{J}$ -vectors with opposite values of  $J_3$ , corresponding to opposite intensity distributions between the two paths in the interferometer. The approximate  $\mathbf{J}$ -vectors of this superposition can be found by noting that the classical limit of eq.(5), where operators are replaced by real numbers, requires that  $J_2 = 0$  and that either  $J_3$  itself or the factor before  $J_3$  must be zero. For  $\eta > 1$ , the classical limit permits solutions with  $J_3 \neq 0$ . From these solutions, the approximate

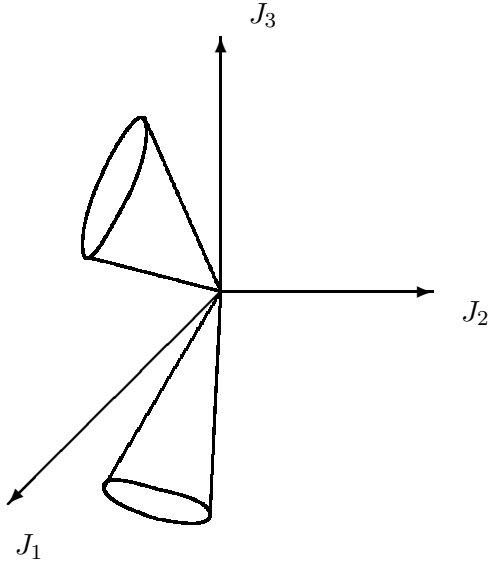


Figure 3: Morning state ( $1 < \eta < 2$ ). The state describes a quantum superposition of  $\hat{J}_2$ -squeezed states with squared squeezing factors greater than two.

average of  $\hat{J}_1$  can be derived as

$$\langle \hat{J}_1 \rangle \approx \frac{N}{2} \left( \frac{2}{\eta} - 1 \right). \quad (8)$$

In the classical limit, the two  $\mathbf{J}$ -vectors that solve eq.(5) have opposite  $J_3$  values of  $\pm N\sqrt{\eta-1}/\eta$ . The amount of squeezing can then be determined by linearizing the squeezing relation (5) around the corresponding points in the  $J_1$ - $J_3$  plane. The approximate  $\hat{J}_2$ -squeezing thus obtained is described by

$$\Delta J_2^2 \approx \frac{\eta-1}{\eta} \frac{N}{4}, \quad (9)$$

indicating that the  $\hat{J}_2$ -squeezing continuously drops back towards the shot noise limit of  $N/4$  as  $\eta$  increases. A representation of the superposition state obtained at  $1 < \eta < 2$ , just after the transition from squeezing to quantum superpositions, is shown in fig. 3. Since the components of the quantum superposition are still in the ‘eastern’ half of the  $\mathbf{J}$ -sphere, we might call this the morning state of our quantum interference scheme.

## 5 Noon states: maximal path entanglement

A special point is reached at  $\eta = 2$ , where the average value of  $\hat{J}_1$  is zero. At this operating point, the  $N$ -photon state is given by a superposition of two squeezed states centered around the poles of the  $\mathbf{J}$ -sphere at  $J_3 = \pm N/2$ . The squeezed noise distribution at high  $N$  is approximately given by  $\delta J_2^2 = N/8$  and  $\delta J_1^2 = N/2$ . In the  $\hat{J}_3$ -basis de-

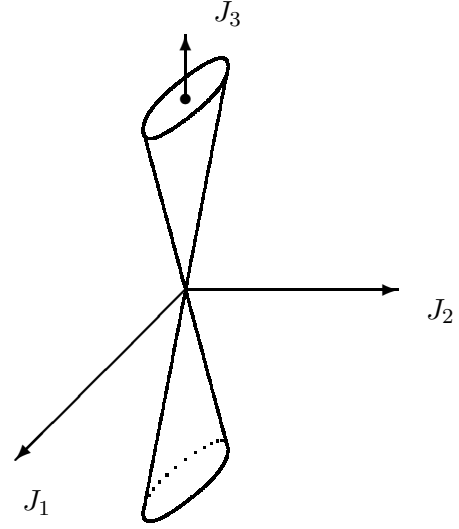


Figure 4: Noon state ( $\eta = 2$ ). Since the quantum superposition between states at  $J_3 = \pm N/2$  is squeezed by a squared factor of two, the noon state fidelity at high  $N$  is 94.3 %.

scribing the photon number distribution between the paths inside the interferometer, this level of squeezing requires only a small addition of the  $|N-2, 2\rangle$  and the  $|2, N-2\rangle$  states to the ideal noon state,

$$|\eta = 2\rangle \approx \left(\frac{2}{9}\right)^{1/4} \left( |N; 0\rangle + \frac{1}{3\sqrt{2}} |N-2; 2\rangle + \frac{1}{3\sqrt{2}} |2; N-2\rangle + |0; N\rangle \right). \quad (10)$$

Thus the overlap between this approximate noon state and an ideal noon state is given by

$$|\langle \text{NOON} | \eta = 2 \rangle|^2 \approx \sqrt{\frac{8}{9}} \approx 0.943. \quad (11)$$

At  $\eta = 2$ , the interference between down-converted light and laser light therefore results in a noon state with a fidelity of 94.3% in the limit of high photon number  $N$  [5]. Significantly, the generation of this state does not require any post-selection conditions and results in a phase sensitivity that is only slightly lower than the Heisenberg limit. A representation of the approximate noon state state is shown in fig. 4. Since the upper component of the quantum superposition has now reached the zenith, the noon state terminology appropriately characterizes this state in the new context of our quantum interference scheme.

## 6 Afternoon states: from cat-states to kitten-states

As  $\eta$  increases beyond two, the squeezing effects quickly become negligible and the quantum states can be described directly by a superposition of two classically coherent states, where all of the photons are in either one or the other of two non-orthogonal optical modes. The  $\mathbf{J}$ -vectors of these two modes are approximately given by the results derived in section 4. In particular, the average value of  $\hat{J}_1$  is given by eq.(8), and the values of  $\hat{J}_3$  are given by  $J_3 = \pm N\sqrt{\eta-1}/\eta$ . A representation of a state with  $\eta > 2$  is shown in fig. 5. As  $\eta$  increases, the two  $\mathbf{J}$ -vectors describing the non-classical superposition approach each other, drawing closer to the ‘western’ end of the  $\mathbf{J}$ -sphere. It therefore seems appropriate to refer to these states as the afternoon states of our quantum interference scheme.

For sufficiently high  $\eta$ , the two  $\mathbf{J}$ -vectors will again merge into a single state centered around  $J_1 = -N/2$ . As the two states begin to overlap, the quantum superposition has effects on the uncertainty distributions that are very similar to those described for the superpositions of coherent states in the context of continuous variables [8]. In particular, the states obtained for even  $N$  are positive superpositions and therefore exhibit  $J_2$ -squeezing, similar to the dawn states of section 4, while the states for odd  $N$  are nega-

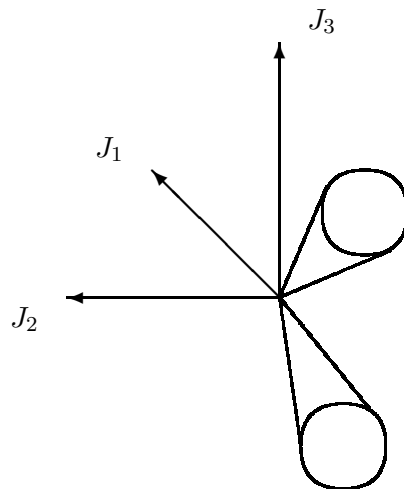


Figure 5: Afternoon state ( $\eta > 2$ ).  $\langle \hat{J}_1 \rangle$  is negative and  $\Delta J_2^2 \approx N/4$  corresponds to shot noise.

tive superpositions and therefore exhibit a three fold increase in  $\Delta J_2^2$  as  $\eta$  goes to infinity. These phenomena become observable as soon as the  $J_3$  values of  $\pm N\sqrt{\eta-1}/\eta$  become smaller than the quantum fluctuations of  $\sqrt{N}/2$ . For high  $N$ , this happens around  $\eta = 4N$ . Fig. 6 illustrates the overlap of the quantum state components at  $\eta > 4N$ . Since these states are close to the ‘western’ horizon, we might refer to them as the evening states of our quantum interference scheme.

## 7 Conclusions

The interference between down-converted light and coherent laser light provides a tunable source of non-classical  $N$ -photon states that has the potential of greatly expanding the range of experimental possibilities in quantum optics. As our analysis shows, the states generated by this method cover both squeezing effects and extremely non-classical superpositions, including a 94% fidelity approximation to the ideal noon state at  $\eta = 2$ . When visualized on the  $\mathbf{J}$ -sphere, the noon state appears as a quantum superposition of a state at the  $J_3 = +N/2$  zenith and

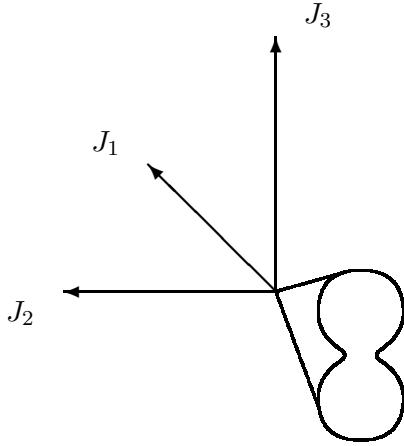


Figure 6: Evening state ( $\eta > 4N$ ). The superposition merges at  $J_1 = -N/2$ , producing interference effects depending on whether the total photon number  $N$  is odd or even.

its mirror image at  $J_3 = -N/2$ . It is therefore tempting to identify the terminology with the position of the  $\mathbf{J}$ -vector describing the upper branch of the quantum superposition, leading to the identification of morning, afternoon, and evening states. Specifically, the linearly squeezed states described in [7] can then be identified as morning states, while a different kind of squeezing observed at the opposite (‘western’) end of the  $\mathbf{J}$ -sphere can be identified as evening states. It is thus possible to connect the noon state terminology with an intuitive image of the quantum statistics associated with the respective non-classical states.

## Acknowledgements

This work was partially supported by the CREST program of the Japan Science and Technology Agency, JST, and the Grant-in-Aid program of the Japanese Society for the Promotion of Science, JSPS.

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